

# Dynamic Linkages for Multivariate Distributions with Given Nonoverlapping Multivariate Marginals

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One of the most useful tools for handling multivariate distributions with given *univariate* marginals is the copula function. Using it, any multivariate distribution function can be represented in a way that emphasizes the separate roles of the marginals and of the dependence structure. Li *et al.* (1996) introduced an analogous tool, called linkage, which is useful for handling multivariate distributions with given *multivariate* marginals. The goal of the present paper is to introduce a new kind of linkage, called the *dynamic linkage*, which can usefully handle multivariate life distributions (that is, distributions of non-negative random variables) by taking advantage of the time dynamics of the underlying lifetimes. Like the linkages of Li *et al.* (1996), the new dynamic linkage can be used for the study of multivariate distributions with given multivariate marginals by emphasizing the separate roles of the dependence structure *among* the given multivariate marginals and the dependence structure *within* each of the nonoverlapping marginals. Preservation of some setwise positive dependence properties, from the dynamic linkage function  $L$  to the joint distribution  $F$  and vice versa, are studied. When two different distribution functions are associated with the same dynamic linkage (that is, have the same setwise dependence structure) we show that the cumulative hazard order among the corresponding multivariate marginal distributions implies an overall stochastic dominance between the two underlying distribution functions. © 1999 Academic Press

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## 1. INTRODUCTION

One of the most useful tools for handling multivariate distributions with given *univariate* marginals is the copula function. Using it, any multivariate distribution function can be represented in a way that emphasizes the separate roles of the marginals and of the dependence structure. Li *et al.* (1996) introduced an analogous tool, called linkage, which is useful for handling multivariate distributions with given *multivariate* marginals. In the present paper we will refer to the linkage of Li *et al.* (1996) as the *standard linkage*.

The goal of the present paper is to introduce a new kind of linkage, called the *dynamic linkage*, which can usefully handle multivariate life distributions (that is, distributions of non-negative random variables) by taking advantage of the time dynamics of the underlying lifetimes. Like the standard linkage, the new dynamic linkage can be used for the study of multivariate distributions with given multivariate marginals by emphasizing the separate roles of the dependence structure *among* the given multivariate marginals and the dependence structure *within* each of the nonoverlapping marginals.

The dynamic linkage function, like the standard linkage, is particularly useful when not all the interrelationships among the random variables are equally important, but rather only the relationships among certain non-overlapping sets of random variables (that is, random vectors) are relevant. The need to study relationships among random vectors arises naturally in a variety of circumstances (see, for example, Chhetry *et al.* (1989) and Block and Fang (1990)). For example, in a complex engineering system, the relationship among the subsystems can be considered in the framework of this paper, even if the dependence structure within the subsystems is not entirely well understood. Similarly, in a Jackson queueing network (see, for example, Section 5 in Daduna and Szekli (1995)), the relationship among vectors of queue lengths at different time points can be considered in the framework of this paper, even if the dependence structure within each such vector of queue lengths, corresponding to any particular time point, is not entirely well understood. Additionally, a framework for studying vector dependencies may lead to further understanding of complicated multivariate distributions.

The approach of the present paper is similar to the approach of Li *et al.* (1996). That is, given a  $(\sum_{i=1}^{\ell} n_i)$ -dimensional life distribution function  $F$ , with the (possibly multivariate) marginal distributions  $F_1, F_2, \dots, F_{\ell}$  of dimensions  $n_1, n_2, \dots, n_{\ell}$ , respectively, we associate with  $F$  the so called dynamic linkage  $L$  which contains the information regarding the dependence structure among the underlying random lifetimes. The dependence structure within the random vectors is not included in  $L$ . A difference between the

present work and the work of Li *et al.* (1996) is that the new dynamic linkage function takes advantage of the fact that the studied random variables are lifetimes (that is, are non-negative), and thus they can be studied through the total hazard construction and by the cumulative hazard order. An important distinction between the dynamic linkage and the previous standard linkage is that the dynamic linkage is invariant with respect to the order of the random variables within each marginal random vector (see the discussion after Remark 3.9).

After giving some preliminaries we give the definition of the dynamic linkage function in Section 3. Preservation of some setwise positive dependence properties (in the sense of Chhetry *et al.* (1989), Joag-Dev *et al.* (1983) Chhetry *et al.* (1986)) from the dynamic linkage function  $L$  to the joint distribution  $F$  and vice versa is studied in Section 4. In some applications two different  $(\sum_{i=1}^{\ell} n_i)$ -dimensional distribution functions may be associated with the same dynamic linkage function (that is, have the same setwise dependence structure). In Section 5 we show that, in such a case, a strong stochastic dominance order, called the cumulative hazard order (see, for example, Shaked and Shanthikumar (1994, p. 127)), among the corresponding multivariate marginal distributions, implies an overall stochastic dominance between the two underlying  $(\sum_{i=1}^{\ell} n_i)$ -dimensional distribution functions.

In this paper “increasing” and “decreasing” stand, respectively, for “non-decreasing” and “nonincreasing.”

## 2. SOME PRELIMINARIES

### 2.1. The Total Hazard Construction

Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be a non-negative random vector with an absolutely continuous distribution function. It will be helpful to think about  $(X_1, X_2, \dots, X_n)$  as the lifetimes of  $n$  components, labeled  $1, 2, \dots, n$ , that make up some system. Consider a typical “history” of  $\mathbf{X}$  at time  $x \geq 0$ , which is of the form

$$h_x = \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\bar{I}} > x\mathbf{e}\}, \quad 0\mathbf{e} \leq \mathbf{x}_I \leq x\mathbf{e}, \quad I \subseteq \{1, 2, \dots, n\}, \quad (2.1)$$

where for each  $n$ -dimensional vector  $\mathbf{x}$  we denote by  $\mathbf{x}_I$  the sub-vector  $(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ , for  $I = (i_1, i_2, \dots, i_k) \subseteq \{1, 2, \dots, n\}$ , and where  $\mathbf{e}$  is a vector of ones of the proper dimension. Note that in (2.1)  $I$  can be thought of as the set of components that have already failed by time  $x$  (with failure times  $\mathbf{x}_I$ ) and  $\bar{I}$  is the set of components that are still alive at time  $x$ .

Given the history  $h_x$  as in (2.1), let  $m \in \bar{I}$  be a component that is still alive at time  $x$ . Its multivariate conditional hazard rate, at time  $x$ , is defined as

$$\lambda_{m|I}(x | \mathbf{x}_I) = \lim_{\Delta x \downarrow 0} \frac{1}{\Delta x} P\{x < X_m \leq x + \Delta x | \mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\bar{I}} > x\mathbf{e}\}, \quad (2.2)$$

where, of course,  $x \geq \max_{i \in I} x_i$ ,  $\min_{i \in I} x_i \geq 0$ , and  $I \subseteq \{1, 2, \dots, n\}$ . As long as the item is alive it accumulates hazard at the rate of  $\lambda_{m|I}(x | \mathbf{x}_I)$  at time  $x$ . If  $I = \{i_1, i_2, \dots, i_k\}$  and  $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_k}$ , then the cumulative hazard of component  $m \in \bar{I}$  at time  $x \geq x_{i_k}$  is

$$\begin{aligned} & \Psi_{m|i_1, i_2, \dots, i_k}(x | x_{i_1}, x_{i_2}, \dots, x_{i_k}) \\ &= \int_0^{x_{i_1}} \lambda_{m|\emptyset}(x | \mathbf{x}_{\emptyset}) du + \sum_{j=2}^k \int_{x_{i_{j-1}}}^{x_{i_j}} \lambda_{m|i_1, i_2, \dots, i_{j-1}}(u | x_{i_1}, x_{i_2}, \dots, x_{i_{j-1}}) du \\ &+ \int_{x_{i_k}}^x \lambda_{m|i_1, i_2, \dots, i_k}(u | x_{i_1}, x_{i_2}, \dots, x_{i_k}) du. \end{aligned} \quad (2.3)$$

The total hazard accumulated by the random failure time  $X_m$ , given that  $X_m$  was the time of the  $k$ th failure, and that the previous failure times were  $X_{i_1}, X_{i_2}, \dots, X_{i_{k-1}}$ , is  $\Psi_{m|i_1, i_2, \dots, i_{k-1}}(X_m | X_{i_1}, X_{i_2}, \dots, X_{i_{k-1}})$ . Note that the functions  $\Psi_{\cdot|\cdot}$  are determined by the multivariate conditional hazard rate functions  $\lambda_{\cdot|\cdot}$  (see (2.3)), which, in turn, are determined by the distribution function  $F$  of  $\mathbf{X}$  (see (2.2)).

Consider the transformation  $\Psi_F: [0, \infty)^n \rightarrow [0, \infty)^n$  (which depends on  $F$ ) defined as follows. For any vector  $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$  define  $(i_1, i_2, \dots, i_n)$  by letting  $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$  be the increasing rearrangement of  $(x_1, x_2, \dots, x_n)$ . Now define

$$\begin{aligned} & \Psi_F(x_1, x_2, \dots, x_n) \\ &= (\Psi_{i_1|\emptyset}(x_{i_1}), \Psi_{i_2|i_1}(x_{i_2} | x_{i_1}), \dots, \Psi_{i_n|i_1, i_2, \dots, i_{n-1}}(x_{i_n} | x_{i_1}, x_{i_2}, \dots, x_{i_{n-1}})), \end{aligned} \quad (2.4)$$

for all  $(x_1, x_2, \dots, x_n)$  in the support of  $\mathbf{X}$ . In the following lemma it is shown that the total hazards accumulated by the random failure times  $X_i$ 's are independent standard (that is, mean one) exponential random variables. This lemma follows from more general results of Aalen and Hoem (1978, Section 4.5), Kurtz (1980, Theorem 6.19(b)), and Jacobsen (1982, Proposition 2.2.11).

LEMMA 2.1. *Let  $X_1, X_2, \dots, X_n$  be  $n$  random variables with an absolutely continuous distribution function  $F$ . Define*

$$(E_1, E_2, \dots, E_n) = \Psi_F(X_1, X_2, \dots, X_n). \quad (2.5)$$

*Then  $E_1, E_2, \dots, E_n$ , are independent standard exponential random variables.*

By “inverting”  $\Psi_F$  we can express the  $X_i$ ’s as functions of the independent standard exponential random variables  $E_1, E_2, \dots, E_n$ . This is the motivation for the following *total hazard construction*.

The idea of the construction is as follows. The components which have the random lifetimes  $X_1, X_2, \dots, X_n$  accumulate hazard as long as they are alive with the rates given in (2.2). Each one of them dies when its accumulated hazard crosses a random threshold. The random thresholds are independent standard exponential random variables  $E_1, E_2, \dots, E_n$ . Thus, by continuously comparing the accumulated hazards to the independent exponential random thresholds it is possible to determine the times in which the accumulated hazards cross the respective thresholds, and these times have the desired distribution function of  $(X_1, X_2, \dots, X_n)$ . From this heuristic description it is seen that the multivariate conditional cumulative hazard functions, given in (2.3), are the only parameters that determine the distribution function of the generated random variables. This is not surprising because it is well-known that indeed the multivariate conditional cumulative hazard functions determine the joint distribution function of the underlying random variables (see, for example, Norros (1986) or Shaked and Shanthikumar (1987)).

Formally, let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be a non-negative random vector with an absolutely continuous distribution function  $F$ . Given the functions  $\Psi_{\cdot|\cdot}(\cdot|\cdot)$  that are associated with  $\mathbf{X}$ , as described in (2.3), we will describe now how to generate a random vector  $\hat{\mathbf{X}} = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$  such that  $\hat{\mathbf{X}} =_{\text{st}} \mathbf{X}$ , where “ $=_{\text{st}}$ ” denotes equality in law. Let  $E_1, E_2, \dots, E_n$  be independent standard exponential random variables.

First we determine the identity  $i_1$  of the component that fails first and its time of failure  $\hat{X}_{i_1}$ . This is determined by

$$\hat{X}_{i_1} = \min\{\tilde{X}_1^1, \tilde{X}_2^1, \dots, \tilde{X}_n^1\},$$

where

$$\tilde{X}_m^1 = \min\{x \geq 0 : \Psi_{m|\varnothing}(x|\varnothing) \geq E_m\}, \quad m = 1, 2, \dots, n,$$

and  $i_1$  is the index of the smallest  $\tilde{X}_m^1$ .

Now suppose that  $\hat{X}_{i_1}, \hat{X}_{i_2}, \dots, \hat{X}_{i_{k-1}}$  have already been determined. Let  $I = \{i_1, i_2, \dots, i_{k-1}\}$  and denote  $\tilde{I} = \{m_1, m_2, \dots, m_{n-k+1}\}$ . The identity  $i_k$  of

the component that is the  $k$ th one to fail and its failure time  $\hat{X}_{i_k}$  are determined by

$$\hat{X}_{i_k} = \min\{\tilde{X}_{m_1}^k, \tilde{X}_{m_2}^k, \dots, \tilde{X}_{m_{n-k+1}}^k\},$$

where here, for  $m \in \bar{I}$ ,

$$\tilde{X}_m^k = \min\{x \geq \hat{X}_{i_{k-1}} : \Psi_{m|i_1, i_2, \dots, i_{k-1}}(x | \hat{X}_{i_1}, \hat{X}_{i_2}, \dots, \hat{X}_{i_{k-1}}) \geq E_m\},$$

and  $i_k$  is the index of the smallest  $\tilde{X}_m^k$ ,  $m \in \bar{I}$ .

The transformation described above, which maps  $(E_1, E_2, \dots, E_n)$  to  $(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n)$ , will be denoted by  $\Psi_F^*$ ; that is,

$$(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n) \equiv \Psi_F^*(E_1, E_2, \dots, E_n). \quad (2.6)$$

It can be shown that indeed

$$(\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n) =_{\text{st}} (X_1, X_2, \dots, X_n); \quad (2.7)$$

see, for example, Shaked and Shanthikumar (1987). In fact, if  $F$  is absolutely continuous then

$$\Psi_F^* \Psi_F(X_1, X_2, \dots, X_n) =_{\text{a.s.}} (X_1, X_2, \dots, X_n), \quad (2.8)$$

where “ $=_{\text{a.s.}}$ ” denotes almost sure equality; see, for example, Norros (1986).

*Remark 2.2.* The assumption of absolute continuity is not really needed for the validity of the total hazard construction. This is because the functions  $\Psi_{\cdot|\cdot}(\cdot|\cdot)$ , given in (2.3), can also be defined by means of logarithms of conditional survival probabilities, and then the assumption of absolute continuity is not used. For example, no continuity assumptions are needed for the validity of (2.7). In this paper we have made this assumption for the sake of simplicity.

## 2.2. The Supportive Lifetimes Property

Let  $X_1, X_2, \dots, X_n$ , be  $n$  random variables with a joint distribution function  $F$ . In general,  $\Psi_F^*(e_1, e_2, \dots, e_n)$  is not necessarily increasing in  $(e_1, e_2, \dots, e_n) \in [0, \infty)^n$ . The property of supportive lifetimes, which is described below, is a condition under which  $\Psi_F^*(e_1, e_2, \dots, e_n)$  is increasing in  $(e_1, e_2, \dots, e_n) \in [0, \infty)^n$ .

Intuitively, the non-negative random variables  $X_1, X_2, \dots, X_n$  (or the random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ ) are said to have the *supportive lifetimes SL property* if surviving components associated with  $\mathbf{X}$  get weaker when other components fail.

More formally, let  $h_x$  and  $h'_x$  be two histories as described in (2.1). If each component that has already failed by time  $x$  in  $h_x$  has also already failed by time  $x$  in  $h'_x$ , and the failures in  $h'_x$  have occurred not later than the corresponding failures in  $h_x$ , then we say that  $h'_x$  is more severe than  $h_x$ , and we denote it by  $h_x \leq h'_x$  (see Shaked and Shanthikumar (1991) for an explicit mathematical description of the severeness order among histories). For any history  $h_x = \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\bar{I}} > x\mathbf{e}\}$  let us denote the total hazard accumulated by  $X_m$  up to time  $x$  given  $h_x$  by

$$Q_m(h_x) \equiv \Psi_{m|I}(x | \mathbf{x}_I), \quad (2.9)$$

where, of course,  $m$  is a component that has not failed by time  $x$  in history  $h_x$ ; that is,  $m \in \bar{I}$ . If

$$Q_m(h_x) \leq Q_m(h'_x) \quad \text{whenever} \quad h_x \leq h'_x, \quad (2.10)$$

where  $m$  is any component that has not failed by time  $x$  in history  $h'_x$ , then  $\mathbf{X}$  is said to have the property of supportive lifetimes (SL) (see, for example, Shaked and Shanthikumar (1990)). When  $F$  is the distribution function of  $\mathbf{X}$ , and  $\mathbf{X}$  has the above property, then we simply say that  $F$  is SL.

The following result is essentially shown in Norros (1986).

**LEMMA 2.3.** *Let  $X_1, X_2, \dots, X_n$  be  $n$  random variables. If  $X_1, X_2, \dots, X_n$  are SL then  $\Psi_F^*(e_1, e_2, \dots, e_n)$  is increasing in  $(e_1, e_2, \dots, e_n) \in [0, \infty)^n$ .*

### 2.3. Copulas

The dynamic linkage (which is introduced in Section 3 below), as the standard linkage of Li *et al.* (1996), can be viewed as a multivariate extension of a copula. In this section we recall the definition and some basic properties of copulas. Most of the results that are reviewed in this subsection will be extended in the sequel. The review in this subsection repeats the one in Li *et al.* (1996).

The copula (as named by Sklar (1959), or the uniform representation as named by Kimeldorf and Sampson (1975), or the dependence function as named by Deheuvels (1978)) is one of the most useful tools for handling multivariate distributions with given univariate marginals  $F_1, F_2, \dots, F_\ell$ .

Formally, a copula  $C$  is a distribution function, defined on  $[0, 1]^\ell$ , with uniform marginals. Given a copula  $C$ , if one defines

$$F(x_1, x_2, \dots, x_\ell) = C(F_1(x_1), F_2(x_2), \dots, F_\ell(x_\ell)), \quad (x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell, \quad (2.11)$$

then  $F$  is a multivariate distribution with univariate marginals  $F_1, F_2, \dots, F_\ell$ . Given a continuous  $F$ , with marginals  $F_1, F_2, \dots, F_\ell$ , there corresponds to it a unique copula that can be constructed as follows:

$$C(u_1, u_2, \dots, u_\ell) = F[F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_\ell^{-1}(u_\ell)], \\ (u_1, u_2, \dots, u_\ell) \in [0, 1]^\ell. \quad (2.12)$$

Note that different multivariate distributions  $F$  may have the same copula. Most of the multivariate dependence structure properties of  $F$  are contained in the copula function, which is independent of the marginals, and which is, in general, easier to handle than the original  $F$ .

We now list some positive dependence properties that are inherited by  $F$  from the corresponding copula. The random vector  $\mathbf{X} = (X_1, X_2, \dots, X_\ell)$  (or its distribution function) is said to be positively upper orthant dependent (PUOD) if

$$P\{X_1 > x_1, X_2 > x_2, \dots, X_\ell > x_\ell\} \geq \prod_{i=1}^{\ell} P\{X_i > x_i\}, \quad (x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell.$$

It is said to be positively lower orthant dependent (PLOD) if

$$P\{X_1 \leq x_1, X_2 \leq x_2, \dots, X_\ell \leq x_\ell\} \geq \prod_{i=1}^{\ell} P\{X_i \leq x_i\}, \quad (x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell$$

(see, for example, Lehmann (1966) or Shaked and Shanthikumar (1994, Subsection 4.G.1)). It is said to be associated if

$$\text{Cov}(g(\mathbf{X}), h(\mathbf{X})) \geq 0, \quad (2.13)$$

for all increasing functions  $g$  and  $h$  for which the covariance is defined (see, for example, Esary *et al.* (1967) or Barlow and Proschan (1975)). Finally,  $\mathbf{X}$  (or its distribution function) is said to be positively dependent by mixtures (PDM) if the joint distribution function  $F$  of  $\mathbf{X}$  can be written as

$$F(x_1, x_2, \dots, x_\ell) = \int_{\Omega} \prod_{i=1}^{\ell} G^{(\mathbf{w})}(x_i) dH(\mathbf{w}),$$

where  $\Omega$  is a subset of a finite-dimensional Euclidean space,  $\{G^{(\mathbf{w})}, \mathbf{w} \in \Omega\}$  is a family of univariate distribution functions, and  $H$  is a distribution function



on  $\Omega$  (see Shaked (1977)). Note that if  $\mathbf{X}$  is PDM then  $X_1, X_2, \dots, X_\ell$  have a permutation symmetric distribution function. The following results are well-known. Recall that a random variable (or vector)  $\mathbf{X}$  is said to be stochastically smaller than the random variable (or vector of the same dimension)  $\mathbf{Y}$  if  $E\phi(\mathbf{X}) \leq E\phi(\mathbf{Y})$  for all real increasing functions  $\phi$  for which the expectations are defined; this relationship will be denoted below by  $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$ .

**PROPOSITION 2.4** (Marshall, 1996). *Let  $C$  be a copula, and let  $F$  be defined as in (2.11).*

- (i) *If  $C$  is PUOD [PLOD] then  $F$  is PUOD [PLOD].*
- (ii) *If  $C$  is associated then  $F$  is associated.*
- (iii) *If  $C$  is PDM, and if  $F_1, F_2, \dots, F_\ell$  of (2.11) are all equal, then  $F$  is PDM.*

**PROPOSITION 2.5** (Scarsini, 1988). *Let  $\mathbf{X} = (X_1, X_2, \dots, X_\ell)$  and  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_\ell)$  have the same copula (as defined in (2.12)). If  $X_i \leq_{\text{st}} Y_i$ ,  $i = 1, 2, \dots, \ell$ , then  $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$ .*

### 3. DYNAMIC LINKAGES

Let  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  be  $\ell$  non-negative random vectors of dimensions  $n_1, n_2, \dots, n_\ell$ , respectively. We do not necessarily assume that the  $\mathbf{X}_i$ 's are independent. Let  $F_i$  be the (marginal)  $n_i$ -dimensional distribution of  $\mathbf{X}_i$ ,  $i = 1, 2, \dots, \ell$ , and let  $F$  be the joint distribution of  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  which is, of course, of dimension  $\sum_{i=1}^\ell n_i$ . For  $i = 1, 2, \dots, \ell$ , let the transformation  $\Psi_{F_i}: [0, \infty)^{n_i} \rightarrow [0, \infty)^{n_i}$  be defined as in (2.4). Then, by (2.5), if  $F_i$  is absolutely continuous, then the vector  $\mathbf{E}_i = \Psi_{F_i}(\mathbf{X}_i)$  is a vector of  $n_i$  independent standard exponential random variables. However, since the  $\mathbf{X}_i$ 's are not necessarily independent, it follows that the  $\mathbf{E}_i$ 's are not necessarily independent. The joint distribution  $L$  of

$$(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell) = (\Psi_{F_1}(\mathbf{X}_1), \Psi_{F_2}(\mathbf{X}_2), \dots, \Psi_{F_\ell}(\mathbf{X}_\ell)) \quad (3.1)$$

will be called the *dynamic linkage* corresponding to  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$ . In general, a dynamic linkage is any joint distribution function of  $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell$ , where the corresponding dimensions are  $n_1, n_2, \dots, n_\ell$ , and each  $\mathbf{E}_i$  consists of independent standard exponential random variables.

Note that different multivariate distribution functions (with marginals of dimensions  $n_1, n_2, \dots, n_\ell$ ) may have the same dynamic linkage. Most of the information, regarding the multivariate dynamic dependence structure

properties *among* the  $\mathbf{X}_i$ 's is contained in the dynamic linkage function, which is independent of the marginals, and which may be easier to handle than the original  $F$ . Note that the dynamic linkage function is not expected to contain any information regarding the dynamic dependence properties *within* each of the  $\mathbf{X}_i$ 's. This information is contained in the  $m_i$ -dimensional functions  $\Psi_{F_i}$ , and it is erased when we transform the vector  $\mathbf{X}_i$ , of dependent variables, into the vector  $\mathbf{E}_i$ , of independent standard exponential random variables, by  $\mathbf{E}_i = \Psi_{F_i}(\mathbf{X}_i)$ . Thus, the dynamic linkage function can be useful when one is interested in studying the dependence properties *among* the  $\mathbf{X}_i$ 's, separate from the dependence properties *within* the  $\mathbf{X}_i$ 's.

If  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  have the joint distribution  $F$ , and if  $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell$  have the joint distribution  $L$ , where  $L$  is the dynamic linkage corresponding to  $F$ , then it is not hard to show, using (2.8), that  $(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \dots, \hat{\mathbf{X}}_\ell)$  defined by

$$(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \dots, \hat{\mathbf{X}}_\ell) \equiv (\Psi_{F_1}^*(\mathbf{E}_1), \Psi_{F_2}^*(\mathbf{E}_2), \dots, \Psi_{F_\ell}^*(\mathbf{E}_\ell)), \quad (3.2)$$

is such that

$$(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \dots, \hat{\mathbf{X}}_\ell) =_{\text{st}} (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell). \quad (3.3)$$

The following lemma is obvious.

**LEMMA 3.1.** *Let  $L_1$  and  $L_2$  be two dynamic linkages of the same corresponding dimensions  $n_1, n_2, \dots, n_\ell$ , and let  $\alpha \in [0, 1]$ . Then  $\alpha L_1 + (1 - \alpha) L_2$  is also a dynamic linkage with the same corresponding dimensions. That is, a mixture of dynamic linkages with the same corresponding dimensions is a dynamic linkage of the same dimensions.*

We now give some examples of typical dynamic linkages.

**EXAMPLE 3.2.** Consider the case in which  $\ell = 2$  and  $n_1 = n_2 = 2$ . Explicitly, we are given now two bivariate marginal distribution functions  $F_1$  and  $F_2$ , say. A dynamic linkage in this case is a 4-dimensional ( $n_1 + n_2 = 4$ ) distribution function  $L$ , of the random vectors  $(E_{11}, E_{12})$  and  $(E_{21}, E_{22})$ , say, where  $E_{11}$  and  $E_{12}$  are independent standard exponential random variables,  $E_{21}$  and  $E_{22}$  are independent standard exponential random variables, but otherwise  $L$  can be any joint distribution. Let  $(\hat{X}_{11}, \hat{X}_{12})$  and  $(\hat{X}_{21}, \hat{X}_{22})$  be defined as in (3.2), and let  $F$  be their joint distribution function. Thus  $F$  is a distribution function that has the dynamic linkage  $L$  and the bivariate marginals  $F_1$  and  $F_2$ .

For example, a particular dynamic linkage can be obtained by taking  $E_{21} = E_{11}$  and  $E_{22} = E_{12}$ . Then the joint distribution function of

$$((E_{11}, E_{12}), (E_{11}, E_{12})) \quad (3.4)$$

is a dynamic linkage. Another example of a dynamic linkage is the joint distribution function of

$$((E_{11}, E_{12}), (E_{11}, -\log(1 - e^{-E_{12}}))). \quad (3.5)$$

Still another example of a dynamic linkage is the joint distribution function of

$$((E_{11}, E_{12}), (-\log(1 - e^{-E_{11}}), -\log(1 - e^{-E_{12}}))). \quad (3.6)$$

By Lemma 3.1, also mixtures of the dynamic linkages described in (3.4)–(3.6) are dynamic linkages.

**EXAMPLE 3.3.** Let  $W$  and  $Z$  be two independent univariate non-negative random variables. Define  $\mathbf{X} = (X_1, X_2) = ((Z, Z + W), W)$ . So here the random vector  $\mathbf{X}$  consists of one 2-dimensional and one 1-dimensional vectors. It is not hard to see, by working out (3.1), that the dynamic linkage associated with  $\mathbf{X}$  is the joint distribution function  $L$  of  $((E_1, E_2), E_2)$ , where  $E_1$  and  $E_2$  are independent standard exponential random variables. In fact,  $L$  is the dynamic linkage of  $((Z, g(Z) + W), W)$  whenever  $g$  satisfies  $g(z) \geq z$  for all  $z \geq 0$ .

The condition  $g(z) \geq z$  that is imposed in Example 3.3 ensures that  $Z \leq_{\text{a.s.}} g(Z) + W$ . Thus the time-dynamic behavior of  $(Z, g(Z) + W)$  can easily be envisioned by means of the random variables  $Z$  and  $W$ . Roughly speaking, if  $Z$  and  $g(Z) + W$  are the lifetimes of two items, 1 and 2, say, then, under the condition  $g(z) \geq z$ , we have that  $Z$  and  $W$  do not affect the items simultaneously. First  $Z$  determines the lifetime of item 1. After  $Z$  has been determined (and therefore also  $g(Z)$ , which is a time point after  $Z$  by the condition  $g(z) \geq z$ ) then  $W$  plays its role as the added lifetime of item 2. In other words, the collection of events  $\{Z > t\}_{t \geq 0}$  is the same as the collection of events  $\{Z > t, g(Z) + W > t\}_{t \geq 0}$ , and therefore initially, as long as item 1 is alive, one can ignore any information involving the random variable  $W$ . Thinking this way, it is apparent that the dynamic linkage of  $\mathbf{X}$  in Example 3.3 is the distribution function of  $((E_1, E_2), E_2)$ . In fact, it is apparent that  $L$  of Example 3.3 is the dynamic linkage of  $((Z, g(Z, W)), W)$  whenever  $g(z, w)$  is increasing in  $w \geq 0$  for all  $z \geq 0$ , and  $g(z, w) \geq z$  for all  $z \geq 0$  and  $w \geq 0$ . For example,  $L$  is the dynamic linkage of  $((Z, g(Z) + 2W), W)$  when  $g$  satisfies  $g(z) \geq z$  for all  $z \geq 0$ .

In order to illustrate the computations that are involved with a construction of a dynamic linkage we have the following example.

**EXAMPLE 3.4.** Consider the random vector  $((X_{11}, X_{12}), (X_{21}, X_{22}))$  whose survival function is given by

$$\begin{aligned}
& P\{X_{11} > x_{11}, X_{12} > x_{12}, X_{21} > x_{21}, X_{22} > x_{22}\} \\
& \equiv \bar{F}((x_{11}, x_{12}), (x_{21}, x_{22})) \\
& = \frac{1}{1 + x_{11} + x_{12} + x_{21} + x_{22}}, \quad x_{ij} \geq 0, \quad i = 1, 2, \quad j = 1, 2.
\end{aligned}$$

Let  $F_1$  be the marginal distribution function of  $(X_{11}, X_{12})$  whose associated survival function is given by

$$P\{X_{11} > x_1, X_{12} > x_2\} \equiv \bar{F}_1(x_1, x_2) = \frac{1}{1 + x_1 + x_2}, \quad x_j \geq 0, \quad j = 1, 2.$$

The survival function  $\bar{F}_2$  which is associated with the distribution function  $F_2$  of  $(X_{21}, X_{22})$  is identical to  $\bar{F}_1$ .

Consider the bivariate function  $\Phi: [0, \infty)^2 \rightarrow [0, \infty)$  defined by

$$\Phi(x_1, x_2) = \begin{cases} \frac{1}{2} \log(1 + 2x_1), & \text{if } x_1 \leq x_2; \\ \frac{1}{2} \log(1 + 2x_2) + 2 \log\left(\frac{1 + x_1 + x_2}{1 + 2x_2}\right), & \text{if } x_1 > x_2. \end{cases}$$

The transformation  $\Psi_{F_1}: [0, \infty)^2 \rightarrow [0, \infty)^2$  (see (2.4)) is given then by

$$\Psi_{F_1}(x_1, x_2) = (\Phi(x_1, x_2), \Phi(x_2, x_1)), \quad x_1 \geq 0, x_2 \geq 0.$$

The transformation  $\Psi_{F_2}: [0, \infty)^2 \rightarrow [0, \infty)^2$  is identical to  $\Psi_{F_1}$ . Therefore, the dynamic linkage which is associated with the distribution function  $F$  of  $((X_{11}, X_{12}), (X_{21}, X_{22}))$  is the distribution function of

$$\begin{aligned}
& ((E_{11}, E_{12}), (E_{21}, E_{22})) \\
& \equiv ((\Phi(X_{11}, X_{12}), \Phi(X_{12}, X_{11})), (\Phi(X_{21}, X_{22}), \Phi(X_{22}, X_{21}))).
\end{aligned}$$

Note that here, indeed,  $E_{11}$  and  $E_{12}$  are independent of each other, and so are  $E_{21}$  and  $E_{22}$ . However, each of  $E_{11}$  and  $E_{12}$  depends on  $(E_{21}, E_{22})$ . Similarly, each of  $E_{21}$  and  $E_{22}$  depends on  $(E_{11}, E_{12})$ .

The following Example 3.5 will be used in Remark 3.8 below. This example also provides an illustration (in addition to Example 3.4) of the computations involved with a construction of a dynamic linkage.

**EXAMPLE 3.5.** Let  $Z$  and  $W$  be two independent standard exponential random variables. Define  $\mathbf{X} = (X_1, X_2) = ((Z, Z/2 + W/2), W)$ . Let  $F_1$  denote the (bivariate marginal) distribution of  $\mathbf{X}_1$  and let  $F_2$  denote the distribution of  $X_2$ . Of course,  $F_2$  is the univariate standard exponential

distribution function. It is not hard to verify that the density function  $f_1$  of  $\mathbf{X}_1 = (X_{11}, X_{12}) = (Z, Z/2 + W/2)$  is given by

$$f_1(x_1, x_2) = \begin{cases} 2e^{-2x_2}, & 2x_2 > x_1 \geq 0; \\ 0, & \text{otherwise;} \end{cases}$$

and that the survival function of  $\mathbf{X}_1$  is given by

$$\bar{F}_1(x_1, x_2) = \begin{cases} (1 + 2x_2 - x_1) e^{-2x_2}, & 2x_2 > x_1 \geq 0; \\ e^{-x_1}, & x_1 \geq 2x_2 \geq 0. \end{cases}$$

Now, by a lengthy computation we obtain that

$$\Psi_{F_1}(x_1, x_2) = (\Phi_1(x_1, x_2), \Phi_2(x_1, x_2)), \quad 2x_2 > x_1 \geq 0,$$

where

$$\Phi_1(x_1, x_2) = \begin{cases} \log(1 + x_1), & 0 \leq x_1 \leq x_2; \\ \log \left[ \frac{(1 + x_2)x_2}{2x_2 - x_1} \right], & 0 \leq x_2 \leq x_1 < 2x_2; \end{cases}$$

and

$$\Phi_2(x_1, x_2) = \begin{cases} 2x_2 - 2 \log(1 + x_1), & 0 \leq x_1 \leq x_2; \\ 2x_2 - 2 \log(1 + x_2), & 0 \leq x_2 \leq x_1. \end{cases}$$

Thus the dynamic linkage which is associated with the distribution function of  $((X_{11}, X_{12}), X_2)$  is the distribution function of

$$\begin{aligned} & ((\Phi_1(X_{11}, X_{12}), \Phi_2(X_{11}, X_{12})), X_2) \\ &= ((\Phi_1(Z, Z/2 + W/2), \Phi_2(Z, Z/2 + W/2)), W) \\ &= ((Z', W'), W), \end{aligned}$$

where (note that  $X_{11} \leq X_{12}$  if, and only if,  $Z \leq W$ )

$$Z' = \begin{cases} \log(1 + Z), & Z \leq W; \\ \log \left[ \frac{(1 + Z/2 + W/2)(Z/2 + W/2)}{W} \right], & Z \geq W; \end{cases}$$

and

$$W' = \begin{cases} Z + W - 2 \log(1 + Z), & Z \leq W; \\ Z + W - 2 \log(1 + Z/2 + W/2), & Z \geq W. \end{cases}$$

At the first glance it seems that Example 3.5 is very similar to Example 3.3. However, this is not the case. In Example 3.5, if  $Z$  and  $Z/2 + W/2$  are the lifetimes of two items, 1 and 2, say, then the event that items 1 and 2 are alive at time  $t$  (that is,  $\{Z > t, Z/2 + W/2 > t\}$ ) contains some information about the possible survival of item 1 beyond time  $2t$ . In other words, the collection of events  $\{Z > t\}_{t \geq 0}$  is different than the collection of events  $\{Z > t, Z/2 + W/2 > t\}_{t \geq 0}$ . For this reason, unlike as in Example 3.3, one cannot initially ignore the information regarding the random variable  $W$  that is contained in the event  $\{Z > t, Z/2 + W/2 > t\}$ , and the computations become more involved.

It is known that if a random vector  $(X_1, X_2, \dots, X_\ell)$ , with continuous marginals, has the copula  $C$ , then the random vector  $(g_1(X_1), g_2(X_2), \dots, g_\ell(X_\ell))$  has the same copula  $C$  whenever the  $g_i$ 's are strictly increasing real univariate functions, that is, the copula is preserved under strictly increasing univariate transformations. The following result is a time-dynamic analog of this fact.

**THEOREM 3.6.** *Let  $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_\ell) = ((X_{11}, \dots, X_{1n_1}), \dots, (X_{\ell 1}, \dots, X_{\ell n_\ell}))$  be a non-negative random vector with an absolutely continuous distribution function, Let  $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = ((Y_{11}, \dots, Y_{1n_1}), \dots, (Y_{\ell 1}, \dots, Y_{\ell n_\ell}))$  be another random vector such that*

$$\begin{aligned} & ((Y_{11}, \dots, Y_{1n_1}), \dots, (Y_{\ell 1}, \dots, Y_{\ell n_\ell})) \\ &=_{\text{st}} ((g_1(X_{11}), \dots, g_1(X_{1n_1})), \dots, (g_\ell(X_{\ell 1}), \dots, g_\ell(X_{\ell n_\ell}))), \end{aligned}$$

where  $g_i$  is a real univariate function which satisfies  $g_i(0) = 0$ , and is strictly increasing on  $[0, \infty)$ ,  $i = 1, \dots, \ell$ . Then  $\mathbf{X}$  and  $\mathbf{Y}$  have the same dynamic linkage.

*Proof.* Denote the distribution function of  $\mathbf{X}_i = (X_{i1}, \dots, X_{in_i})$  by  $F_i$ , and denote the distribution function of  $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{in_i})$  by  $G_i$ ,  $i = 1, \dots, \ell$ . It is easy to verify from (2.4) that

$$\Psi_{G_i}(x_1, \dots, x_{n_i}) = \Psi_{F_i}(g_i^{-1}(x_1), \dots, g_i^{-1}(x_{n_i})), \quad (x_1, \dots, x_{n_i}) \in [0, \infty)^{n_i},$$

$i = 1, \dots, \ell$ . Therefore

$$\Psi_{G_i}(g_i(X_{i1}), \dots, g_i(X_{in_i})) =_{\text{st}} \Psi_{F_i}(X_{i1}, \dots, X_{in_i}), \quad i = 1, \dots, \ell,$$

and the stated result follows from the definition (3.1) of the dynamic linkage. ■

It is of interest to note that Theorem 3.6 shows that the dynamic linkages remain unchanged when different changes of speed of time are applied to the different  $n_i$ -dimensional marginals. The change of speed that

is applied to the individual components of each  $n_i$ -dimensional marginal must be the same in order to preserve the time dynamics, and in order to apply the total hazard construction *within* each  $n_i$ -dimensional marginal.

*Remark 3.7.* If two random vectors  $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  and  $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$  have the same dynamic linkage  $L$  it does not necessarily follow that they have the same copula  $C$ . This is so because the copula is affected by all the dependencies among the  $\sum_{i=1}^\ell n_i$  underlying random variables, whereas the dynamic linkage is affected only by the dependencies between the  $\ell$  underlying random vectors. To see it, let  $W$  and  $Z$  be two independent univariate random variables as in Example 3.3. Define  $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2) = ((Z, Z + W), W)$  and  $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2) = ((Z, 2Z + W), W)$ . From Example 3.3 it follows that  $\mathbf{X}$  and  $\mathbf{Y}$  have the same dynamic linkage  $L$ , but it is easy to see, for example when  $W$  and  $Z$  are standard exponential random variables, that  $\mathbf{X}$  and  $\mathbf{Y}$  do not have the same copula  $C$ .

In the next two Remarks (3.8 and 3.9 below) we point out two facts regarding the standard linkages of Li *et al.* (1996) and the dynamic linkages. The remarks show that if two random vectors have the same standard (respectively, dynamic) linkages then they need not necessarily have the same dynamic (respectively, standard) linkages. That is, dynamic linkages and standard linkages treat dependencies among random vectors in a different manner.

*Remark 3.8* (Same Standard and Different Dynamic Linkages). Let  $Z$  and  $W$  be independent standard exponential random random variables as in Example 3.5. Consider the random vectors

$$\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2) = ((Z, Z/2 + W/2), W)$$

and

$$\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2) = ((Z, W), W).$$

By Example 3.2 of Li *et al.* (1996) it is seen that  $\mathbf{X}$  and  $\mathbf{Y}$  have the same standard linkages.

Since  $Z$  and  $W$  are independent standard exponential random variables, it follows that the dynamic linkage of  $\mathbf{Y}$  is just the distribution function of  $\mathbf{Y}$ , that is, of  $((Z, W), W)$ . The dynamic linkage of  $\mathbf{X}$  is given in Example 3.5 as the distribution function of  $((Z', W'), W)$ . Since  $W' \neq_{\text{a.s.}} W$  (the notation “ $\neq_{\text{a.s.}}$ ” stands for “not equal almost surely”) it follows that

$$((Z, W), W) \neq_{\text{st}} ((Z', W'), W)$$

(the notation “ $\neq_{\text{st}}$ ” stands for “not equal in law”); that is,  $\mathbf{X}$  and  $\mathbf{Y}$  have different dynamic linkages.

In the next example we use the following notation. For any random variable  $X$  we let  $F_X$  denote its distribution function. It follows that if  $X$  is continuous then  $F_X(X)$  is distributed uniformly on  $[0, 1]$ . Similarly, if the random variables  $X$  and  $Y$  have an absolutely continuous joint distribution function then we denote by  $F_{Y|X}(\cdot|\cdot)$  the conditional distribution of  $Y$  given  $X$ . Thus we have that  $F_{Y|X}(Y|X)$  is uniformly distributed on  $[0, 1]$ .

*Remark 3.9* (Same Dynamic and Different Standard Linkages). Let  $Z$  and  $W$  be independent random random variables (for example, we can take them to be standard exponentials), and let  $g$  be a function such that  $g(z) \geq z$ . Then, as in Example 3.3,  $((Z + W, Z), W)$  and  $((g(Z) + W, Z), W)$  have the same dynamic linkages.

Now, the standard linkage of  $((Z + W, Z), W)$  is the distribution function of  $((U_1, U_2), V)$  where

$$\begin{aligned} U_1 &= F_{Z+W}(Z + W), \\ U_2 &= F_{Z|Z+W}(Z | Z + W), \\ V &= F_W(W). \end{aligned}$$

The standard linkage of  $((g(Z) + W, Z), W)$  is the distribution function of  $((U'_1, U'_2), V')$  where

$$\begin{aligned} U'_1 &= F_{g(Z)+W}(g(Z) + W), \\ U'_2 &= F_{Z|g(Z)+W}(Z | g(Z) + W), \\ V' &= F_W(W). \end{aligned}$$

Obviously the distribution of  $(U'_1, V')$  depends on  $g$ . Therefore, if  $g$  is not the identity function, then  $((Z + W, Z), W)$  and  $((g(Z) + W, Z), W)$  do not have the same standard linkages.

In Example 3.2 of Li *et al.* (1996) it is argued that the standard linkage of  $((Z, g(Z) + W), W)$  is independent of  $g$ . However, it is shown in Remark 3.9 above that the standard linkage of  $((g(Z) + W, Z), W)$  *does* depend on  $g$ . This is because the standard linkage is affected by the order of the random variables in each marginal random vector. On the other hand, the dynamic linkage is invariant with respect to the order of the random variables in each marginal random vector in the sense that if the random variables of a marginal random vector are permuted then the same permutation applies to the corresponding independent exponential random variables that determine the dynamic linkage, but otherwise the dynamic linkage is unaffected. This fact points out a desirable advantage that the dynamic linkage has over the standard linkage. This fact also explains why we can apply Example 3.3 in Remark 3.9 above.



Remarks 3.8 and 3.9 show some essential differences between the standard and the dynamic linkages. It is worthwhile to point out that when the random variables in each marginal vector are almost surely ordered, then these two linkages are essentially the same. More explicitly, consider the  $\ell$  random vectors  $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{in_i})$ ,  $i = 1, 2, \dots, \ell$ . Let  $\mathbf{E}_i = (E_{i1}, E_{i2}, \dots, E_{in_i})$ ,  $i = 1, 2, \dots, \ell$ , denote the vectors of independent standard exponential random variables whose joint distribution is the dynamic linkage associated with  $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$ , and let  $\mathbf{U}_i = (U_{i1}, U_{i2}, \dots, U_{in_i})$ ,  $i = 1, 2, \dots, \ell$ , denote the vectors of independent uniform  $[0, 1]$  random variables whose joint distribution is the standard linkage associated with  $\mathbf{X}$ . If  $X_{i1} \leq X_{i2} \leq \dots \leq X_{in_i}$ , almost surely,  $i = 1, 2, \dots, \ell$ , then the  $E_{ij}$ 's can be obtained from the  $U_{ij}$ 's by the relationships

$$E_{ij} = -\log(1 - U_{ij}), \quad \text{and, vice versa,} \quad U_{ij} = 1 - \exp\{-E_{ij}\}, \quad (3.7)$$

$j = 1, 2, \dots, n_i$ ,  $i = 1, 2, \dots, \ell$ . This fact can be seen by noting that when the components of a random vector are almost surely ordered, then the standard construction and the total hazard construction are the same except for the relationships given in (3.7).

From the above discussion, and from Corollary 3.6 of Li *et al.* (1996), we obtain the following result.

**PROPOSITION 3.10.** *Let  $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_\ell) = ((X_{11}, \dots, X_{1n_1}), \dots, (X_{\ell 1}, \dots, X_{\ell n_\ell}))$  and  $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_\ell) = ((Y_{11}, \dots, Y_{1n_1}), \dots, (Y_{\ell 1}, \dots, Y_{\ell n_\ell}))$  be two random vectors. If  $X_{ij} \leq_{\text{a.s.}} X_{i(j+1)}$  and if  $Y_{ij} \leq_{\text{a.s.}} Y_{i(j+1)}$ ,  $j = 1, 2, \dots, n_i - 1$ ,  $i = 1, 2, \dots, \ell$ , and if  $\mathbf{X}$  and  $\mathbf{Y}$  have the same unique copula, then  $\mathbf{X}$  and  $\mathbf{Y}$  have the same dynamic linkage.*

We believe that in general (that is, without the assumption  $X_{ij} \leq_{\text{a.s.}} X_{i(j+1)}$ ) in Proposition 3.10) if  $\mathbf{X}$  and  $\mathbf{Y}$  have the same unique copula then they have the same dynamic linkage. However, so far we have not been able to show this.

Finally we point out that if two pairs of vectors have different standard (dynamic) linkages, then the corresponding pairs of vectors of order statistics may have the same standard (dynamic) linkages. In order to see it, let  $Z$  and  $W$  be two independent random variables. We know that

$$((Z, W), (Z, W)) \quad \text{and} \quad ((Z, W), (W, Z))$$

have different standard and dynamic linkages. But the vectors of order statistics

$$((\min(Z, W), \max(Z, W)), (\min(Z, W), \max(Z, W)))$$

$$\text{and} \quad ((\min(Z, W), \max(Z, W)), (\min(W, Z), \max(W, Z)))$$

have the same distribution functions, and therefore have the same standard and dynamic linkages.

#### 4. PRESERVATION OF POSITIVE DEPENDENCE PROPERTIES

The following extensions of the PUOD and the PLOD concepts, applied to sets of random variables, were introduced in Chhetry *et al.* (1989) (see also Joag-Dev *et al.* (1983) and Chhetry *et al.* (1986), and a short review in Li *et al.* (1996) which we repeat here). The  $\ell$  random vectors,  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  of dimensions  $n_1, n_2, \dots, n_\ell$ , respectively (or their joint distribution function), are said to be setwise positively upper set dependent (SPUSD) if

$$P \left[ \bigcap_{i=1}^{\ell} \{ \mathbf{X}_i \in U_i \} \right] \geq \prod_{i=1}^{\ell} P[ \mathbf{X}_i \in U_i ],$$

for all upper sets  $U_i$  in  $\mathbb{R}^{n_i}$ ,  $i = 1, 2, \dots, \ell$  (a set of  $U[B]$  is an upper [lower] set in  $\mathbb{R}^\ell$  if  $(x_1, x_2, \dots, x_\ell) \in U[B]$  and  $(s_1, x_2, \dots, x_\ell) \leq [\geq] (y_1, y_2, \dots, y_\ell)$  imply that  $(y_1, y_2, \dots, y_\ell) \in U[B]$ ). The random vectors  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  (or their joint distribution function) are said to be setwise positively lower set dependent (SPLSD) if

$$P \left[ \bigcap_{i=1}^{\ell} \{ \mathbf{X}_i \in B_i \} \right] \geq \prod_{i=1}^{\ell} P[ \mathbf{X}_i \in B_i ],$$

for all lower sets  $B_i$  in  $\mathbb{R}^{n_i}$ ,  $i = 1, 2, \dots, \ell$ . It is not hard to verify that  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  are SPUSD if, and only if,

$$E \left[ \prod_{i=1}^{\ell} g_i(\mathbf{X}_i) \right] \geq \prod_{i=1}^{\ell} E[ g_i(\mathbf{X}_i) ],$$

for all non-negative increasing  $n_i$ -dimensional functions  $g_i$ ,  $i = 1, 2, \dots, \ell$ , for which the expectations exist. Similarly,  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  are SPLSD if, and only if,

$$E \left[ \prod_{i=1}^{\ell} h_i(\mathbf{X}_i) \right] \geq \prod_{i=1}^{\ell} E[ h_i(\mathbf{X}_i) ],$$

for all non-negative decreasing  $n_i$ -dimensional functions  $h_i$ ,  $i = 1, 2, \dots, \ell$ , for which the expectations exist. In particular, when  $\ell = 2$ , then  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are SPUSD if, and only if,

$$\text{Cov}(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)) \geq 0,$$

for all real non-negative increasing functions  $g_1$  and  $g_2$  (of the proper dimensions) for which the covariance is well defined. A similar statement holds also for pairs of SPLSD random vectors.

These setwise positive dependence properties are often inherited from the dynamic linkage by the resulting joint distribution. This is shown in the next result. Note that no continuity assumptions are needed for the validity of this theorem. The proof of the next theorem is omitted since it is very similar to the proof of Theorem 4.1 in Li *et al.* (1996). Recall from (2.10) the definition of supporting lifetimes (SL).

**THEOREM 4.1.** *Let  $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell$  be distributed according to a dynamic linkage  $L$ , and let  $F_1, F_2, \dots, F_\ell$  be  $\ell$  (possibly multivariate) distributions. Let  $F$  be a distribution that has the dynamic linkage  $L$  and marginals  $F_1, F_2, \dots, F_\ell$  (that is,  $F$  is the distribution of*

$$(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) = (\Psi_{F_1}^*(\mathbf{E}_1), \Psi_{F_2}^*(\mathbf{E}_2), \dots, \Psi_{F_\ell}^*(\mathbf{E}_\ell)); \quad (4.1)$$

*see (3.2) and (3.3)). If  $L$  is SPUPD [SPLSD] and if each  $F_i$  is SL, then  $F$  is SPUSD [SPLSD].*

The property of association (see (2.13)) is also often inherited from the dynamic linkage by the resulting joint distribution. This is shown in the next result. Following (2.13), we say that a dynamic linkage  $L$  is associated if  $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell$  have the joint distribution  $L$  and the vector  $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell)$  (of dimension  $\sum_{i=1}^\ell n_i$ ) is associated in the sense that

$$\text{Cov}(g(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell), h(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell)) \geq 0,$$

for all real increasing functions  $g$  and  $h$  (of dimension  $\sum_{i=1}^\ell n_i$ ) for which the covariance is defined. Note that although each  $\mathbf{E}_i$  consists of independent random variables, the whole vector  $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell)$  can be positively dependent because of some positive relationship *among* the  $\mathbf{E}_i$ 's. Again, note that no continuity assumptions are needed for the validity of the next theorem. Again we omit the proof of the next theorem since it is very similar to the proof of Theorem 4.2 in Li *et al.* (1996).

**THEOREM 4.2.** *Let  $L, F_1, F_2, \dots, F_\ell$  and  $F$  be as in Theorem 4.1. If  $L$  is associated, and if each  $F_i$  is SL, then  $F$  is associated.*

Note that in Theorem 4.2 the assumption that each  $F_i$  is SL implies at once that each vector  $\mathbf{X}_i$  is associated from *within* (see, for example, Norros (1986)). The association of the dynamic linkage gives us then the positive dependence (within and among) all of the  $\sum_{i=1}^\ell n_i$  underlying random variables.

Chhetry *et al.* (1989) extended the notion of PDM to the multivariate case as follows. The random vector  $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  (or its distribution function), where each  $\mathbf{X}_i$  is  $n$ -dimensional, is said to be setwise dependent by mixture (SDM) if the joint distribution function  $F$  of  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  has the representation

$$F(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\ell) = \int_{\Omega} \prod_{i=1}^{\ell} G^{(\mathbf{w})}(\mathbf{x}_i) dH(\mathbf{w}),$$

where  $\Omega$  is some subset of a finite-dimensional Euclidean space,  $\{G^{(\mathbf{w})}, \mathbf{w} \in \Omega\}$  is some family of  $n$ -dimensional distribution functions, and  $H$  is a distribution function on  $\Omega$ . Note that if  $\mathbf{X}$  is SDM then  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  all have the same marginal distribution functions.

In the next result it is shown that the property of SDM is inherited from the dynamic linkage by the resulting distribution function, under proper dimensionality conditions combined with the requirement that the marginal distribution functions are all equal. Again, no continuity assumptions are needed for the validity of the next theorem. Also, note that the marginals here are not required to be SL. Once more, we omit the proof of the next theorem since it is very similar to the proof of Theorem 4.3 in Li *et al.* (1996).

**THEOREM 4.3.** *Consider a dynamic linkage  $L$  and let  $F_1 = F_2 = \dots = F_\ell$  be  $\ell$   $n$ -dimensional distribution functions, and consider the distribution  $F$  of the vector defined in (4.1). If  $L$  is SDM then  $F$  is SDM.*

## 5. STOCHASTIC COMPARISONS

In this section we extend Proposition 2.5 to the multivariate case. The extension is similar in spirit to Theorem 5.4 of Li *et al.* (1996), but it is essentially different since the total hazard construction, rather than the standard construction, is used here.

Let  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  and  $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell$  be two sets of (possibly dependent) random vectors. Naturally, it is assumed that  $\mathbf{X}_i$  and  $\mathbf{Y}_i$  have the same dimension,  $n_i$ , say,  $i = 1, 2, \dots, \ell$ . Let  $F$  be the joint distribution of  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$ , with marginals  $F_i$ ,  $i = 1, 2, \dots, \ell$ , and let  $G$  be the joint distribution of  $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell$ , with marginals  $G_i$ ,  $i = 1, 2, \dots, \ell$ . We will assume below that  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  and  $(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$  have the same dynamic linkage  $L$ . That is, we will assume that

$$(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell) = (\Psi_{F_1}(\mathbf{X}_1), \Psi_{F_2}(\mathbf{X}_2), \dots, \Psi_{F_\ell}(\mathbf{X}_\ell))$$

and

$$(\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_\ell) = (\Psi_{G_1}(\mathbf{Y}_1), \Psi_{G_2}(\mathbf{Y}_2), \dots, \Psi_{G_\ell}(\mathbf{Y}_\ell))$$

satisfy

$$(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell) =_{\text{st}} (\mathbf{K}_1 \mathbf{K}_2, \dots, \mathbf{K}_\ell) \quad (5.1)$$

with the common dynamic linkage  $L$ . One would expect in this case, in light of Proposition 2.5, that if  $\mathbf{X}_i \leq_{\text{st}} \mathbf{Y}_i$ ,  $i = 1, 2, \dots, \ell$ , then  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) \leq_{\text{st}} (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$ . It turns out that by just assuming  $\mathbf{X}_i \leq_{\text{st}} \mathbf{Y}_i$ ,  $i = 1, 2, \dots, \ell$ , we were not able to obtain the conclusion  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) \leq_{\text{st}} (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$ . We need to assume a little more; see (5.4) below.

First we recall the definition of the cumulative hazard order (see, for example, Shaked and Shanthikumar (1994, p. 127)). Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be a non-negative random vector with cumulative hazard functions  $Q(\cdot|\cdot)$ 's as defined in (2.9). Let  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$  be another non-negative random vector with cumulative hazard functions  $R(\cdot|\cdot)$ 's that are similarly defined. Then we say that  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  is smaller than  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$  in the cumulative hazard order (denoted  $\mathbf{X} \leq_{\text{ch}} \mathbf{Y}$ ) if

$$Q_m(h'_x) \geq R_m(h_x) \quad \text{whenever} \quad h_x \leq h'_x, \quad (5.2)$$

where  $m$  is any component that has not failed by time  $x$  in history  $h'_x$ . Note that the relation  $\leq_{\text{ch}}$  is not an order in the usual sense. In fact, it is obvious from (2.10) that

$$\mathbf{X} \leq_{\text{ch}} \mathbf{X} \Leftrightarrow \mathbf{X} \quad \text{is SL.} \quad (5.3)$$

In Theorem 4.C.1 of Shaked and Shanthikumar (1994) it is shown that for two non-negative random vectors  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  and  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$  one has

$$\mathbf{X} \leq_{\text{ch}} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{\text{st}} \mathbf{Y}. \quad (5.4)$$

**THEOREM 5.1.** *Let  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell$  and  $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell$  be two sets of (possibly dependent) random vectors. (We assume that  $\mathbf{X}_i$  and  $\mathbf{Y}_i$  have the same dimension,  $n_i$ , say,  $i = 1, 2, \dots, \ell$ ). If  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  and  $(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$  have absolutely continuous distribution functions, and if they have the same dynamic linkage  $L$  (in the sense of (5.1)), and if*

$$\mathbf{X}_i \leq_{\text{ch}} \mathbf{Y}_i, \quad i = 1, 2, \dots, \ell, \quad (5.5)$$

then

$$(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) \leq_{\text{st}} (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell). \quad (5.6)$$

*Proof.* In order to obtain (5.6) we will show that there exist, on the same probability space, random vectors  $(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \dots, \hat{\mathbf{X}}_\ell)$  and  $(\hat{\mathbf{Y}}_1, \hat{\mathbf{Y}}_2, \dots, \hat{\mathbf{Y}}_\ell)$  which satisfy

$$(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \dots, \hat{\mathbf{X}}_\ell) =_{\text{st}} (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell), \quad (5.7)$$

$$(\hat{\mathbf{Y}}_1, \hat{\mathbf{Y}}_2, \dots, \hat{\mathbf{Y}}_\ell) =_{\text{st}} (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell), \quad (5.8)$$

and

$$\hat{\mathbf{X}}_i \leq_{\text{a.s.}} \hat{\mathbf{Y}}_i, \quad i = 1, 2, \dots, \ell. \quad (5.9)$$

The result then follows from a well-known characterization of the order  $\leq_{\text{st}}$  (see, for example, Shaked and Shanthikumar (1994, Theorem 4.B.1)).

Denote the marginal distribution function of  $\mathbf{X}_i$  by  $F_i$ , and denote the marginal distribution function of  $\mathbf{Y}_i$  by  $G_i$ ,  $i = 1, 2, \dots, \ell$ . In order to define  $\hat{\mathbf{X}}_i = (\hat{X}_{i1}, \hat{X}_{i2}, \dots, \hat{X}_{in_i})$  and  $\hat{\mathbf{Y}}_i = (\hat{Y}_{i1}, \hat{Y}_{i2}, \dots, \hat{Y}_{in_i})$ ,  $i = 1, 2, \dots, \ell$  let  $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_\ell)$  have the dynamic linkage  $L$  as its distribution function. Now define, as in (2.6),

$$(\hat{X}_{i1}, \hat{X}_{i2}, \dots, \hat{X}_{in_i}) \equiv \Psi_{F_i}^*(E_{i1}, E_{i2}, \dots, E_{in_i}),$$

and

$$(\hat{Y}_{i1}, \hat{Y}_{i2}, \dots, \hat{Y}_{in_i}) \equiv \Psi_{G_i}^*(E_{i1}, E_{i2}, \dots, E_{in_i}),$$

where  $(E_{i1}, E_{i2}, \dots, E_{in_i}) = \mathbf{E}_i$ ,  $i = 1, 2, \dots, \ell$ . From (2.6) and (2.7) we obtain (5.7) and (5.8). Finally, from (5.5) it is seen (as, for example, in the proof of Theorem 4.C.1 of Shaked and Shanthikumar (1994)) that (5.9) holds. ■

The assumption of absolute continuity in Theorem 5.1 is not really essential; see Remark 2.2.

One may ask whether, under the conditions of Theorem 5.1, it is possible to obtain the conclusion  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) \leq_{\text{ch}} (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$ , which is stronger than (5.6). It turns out that, in general, that is not the case. In order to see it, take  $\mathbf{X}_i = \mathbf{Y}_i$ ,  $i = 1, 2, \dots, \ell$ , in Theorem 5.1, where each  $\mathbf{X}_i$  is SL. Then, by (5.3), we have that (5.5) holds. However, if the conclusion of Theorem 5.1 were  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) \leq_{\text{ch}} (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_\ell)$ , that is,  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell) \leq_{\text{ch}} (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$ , then it would have followed, again by (5.3), that  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  is SL. But in general  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_\ell)$  needs not have this positive dependence property. For example, let  $\mathbf{X}_1 = (W, W + Z)$  and  $\mathbf{X}_2 = V_1 I_{[0, W)} + V_2 I_{[W, \infty)}$ , where  $W, Z, V_1$ , and  $V_2$  are independent non-negative random variables, and where  $V_i$  is an exponential random variable with hazard rate  $\lambda_i$ ,  $i = 1, 2$ , such that  $\lambda_1 > \lambda_2$ . Then  $\mathbf{X}_1 \leq_{\text{ch}} \mathbf{X}_1$  (since  $\mathbf{X}_1$  is SL), and  $\mathbf{X}_2 \leq_{\text{ch}} \mathbf{X}_2$  (for any non-negative univariate random

variable  $X$  we have  $X \leq_{\text{ch}} X$ , but  $((W, W+Z), V_1 I_{[0, W)} + V_2 I_{[W, \infty)}) = (W, W+Z, V_1 I_{[0, W)} + V_2 I_{[W, \infty)})$  is not SL. The latter claim can be seen from the fact that upon the failure of the component with lifetime  $W$ , the hazard rate of  $V_1 I_{[0, W)} + V_2 I_{[W, \infty)}$  jumps down (when the component with that lifetime is still alive).

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